A limiter-based well-balanced discontinuous Galerkin method for shallow-water flows with wetting and drying: One-dimensional case

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Abstract

An important part in the numerical simulation of tsunami and storm surge events is the accurate modeling of flooding and the appearance of dry areas when the water recedes. This paper proposes a new algorithm to model inundation events with piecewise linear Runge-Kutta discontinuous Galerkin approximations applied to the shallow water equations. This study is restricted to the one-dimensional case and shows a detailed analysis and the corresponding numerical treatment of the inundation problem.

The main feature is a velocity based “limiting” of the momentum distribution in each cell, which prevents instabilities in case of wetting or drying situations. Additional limiting of the fluid depth ensures its positivity while preserving local mass conservation. A special flux modification in cells located at the wet/dry interface leads to a well-balanced method, which maintains the steady state at rest. The discontinuous Galerkin scheme is formulated in a nodal form using a Lagrange basis. The proposed wetting and drying treatment is verified with several numerical simulations. These test cases demonstrate the well-balancing property of the method and its stability in case of rapid transition of the wet/dry interface. We also verify the conservation of mass and investigate the convergence characteristics of the scheme.

Keywords: Shallow water equations, Discontinuous Galerkin, Wetting and Drying, Limiter, Well-balanced Scheme

1. Introduction

The shallow water equations are an established model for geoscientific applications such as tsunami or storm surge simulations [see e.g. 4, 26]. Although they are derived under the assumption that vertical velocities are negligible, they are favored for their ability to realistically model large scale horizontal flows with relatively low computational cost. While the discrete representation of the flow field and the propagation of surface waves in the deep ocean is usually well captured, difficulties arise in the coastal area, where water floods originally dry areas or recedes back into the ocean. Mathematically, this is a problem, because the shallow water equations become ill-posed when the fluid depth goes to zero. Therefore, either the computational domain’s boundary must be dynamically adjusted to the area covered by the water body, or one needs to introduce a special treatment of wetting and drying events into the numerical scheme.

Although the Lagrangian approach of a moving mesh is known to result in accurate solutions, its implementation is difficult, especially in case of complex bathymetry, and only applications to simple flow configurations have been reported in the literature [3, 20]. Thus, in geophysical problems it turned out to be advantageous to use meshes which do not necessarily align with the wet/dry interface. In this Eulerian approach one has to ensure the positivity of the fluid depth, the proper treatment of dry cells and the discrete representation of steady states (well-balancing). Furthermore, the scheme must stably deal with...
a possibly ill-conditioned velocity near the wet/dry interface when computations are carried out with fluid depth and momentum as primary variables and the velocity is computed as quotient of both quantities. This is especially the case in Godunov-type schemes, which employ Riemann solvers for the flux computation at discontinuous interfaces in the discrete solution.

There have been various approaches to deal with wetting and drying, most of them using finite volume discretization techniques. In this area, the work of Audusse et al. introduced a general treatment of inundation modeling. Their principle of hydrostatic reconstruction together with a well-balanced discretization of the source term has become widely used and further developed. Although finite volume methods have demonstrated their robustness and can perfectly conserve mass, they also have their shortcomings. Most notably, they only provide cell mean values as solution components and the computation of higher order approximations involves an increasing stencil of cells, which becomes complicated for unstructured grids. On the other hand, continuous finite element methods provide pointwise solutions, and they can relatively easily be extended to higher order by using higher order shape functions. The problem in using finite element methods for advection dominated problems is the continuity condition on the solution, which renders most methods unconditionally unstable, and requires stabilization by added artificial viscosity.

In this context, discontinuous Galerkin (DG) methods are a good compromise. They borrow the conservation property from finite volume methods by using local shape functions which are discontinuous across cell interfaces. Communication between cells is modeled by fluxes as in finite volume methods. Higher-order accuracy can be achieved by using high-order shape functions. Compared to continuous finite elements of the same order, discontinuous Galerkin methods need a higher number of degrees of freedom and have a strict time step restriction, but the locality of their shape functions simplifies adaptivity and the parallelization of the method.

Modeling wetting and drying in shallow water flows using discontinuous Galerkin methods is still a very young research area and mostly restricted to linear elements. Bokhove applied a moving mesh approach with free boundary elements at the wet/dry interface, while other approaches were based on fixed meshes. A nodal flux modification technique was introduced by Gourgue et al. The most common approach is based on slope-limiting. Kesserwani & Liang directly adapted the idea of hydrostatic reconstruction to a Runge-Kutta discontinuous Galerkin (RKDG) discretization by differentiating between fully wet and dry cells. The works in introduced a scaling around the mean within each cell to obtain positivity of the fluid depth as well as mass conservation. Most of these studies concentrated on the positivity of the fluid depth and the well-balancing property, so far.

The ill-conditioned computation of the velocity has only been implicitly dealt with by using restrictive limiters, such as the corrected Minmod-limiter in, or, as in, by setting an upper problem dependent tolerance of the velocity. The common approach of setting the velocity to zero under a certain threshold of the fluid depth alone did not yield satisfactory results in our experiments using the DG method. It leads to arbitrarily small time steps and needs further stabilization as described in Meister & Ortleb using a regularization and modal filtering. Furthermore, additional problem dependent parameters are often introduced.

The problem of stably computing the velocity is addressed in this study in the context of a limiter based treatment of inundation events. The basic idea is that the momentum variable is modified on the basis of the resulting velocity distribution given a fixed (but already limited) distribution of the fluid depth. This results in a stable flux computation which usually involves the computation of the velocity at some point. The general idea is borrowed from finite volume methods, where limiting in other than the primary flow variables often enhances the solution (see van Leer and references therein). In case of compressible gas dynamics this approach can easily ensure that the pressure always remains non-negative and velocity and pressure stay constant across contact discontinuities. For shallow water flows limiting in surface elevation generally makes sense, since this quantity is constant for the steady state at rest. Furthermore, limiting in the velocity instead of momentum has been often employed in these methods. However, in finite volume methods, these limited values are only used for the flux computation at the cell interfaces. In discontinuous Galerkin methods, on the other hand, the solution itself is limited and further used throughout the computations. Therefore, the non-trivial in-cell functional behavior of the velocity, which is the quotient
of two polynomials, cannot be ignored in the limiting process. To the authors’ knowledge this concept of limiting in other than the primary variables has not been thoroughly transferred to discontinuous Galerkin methods, yet.

In addition, the proposed method is fully mass conservative and well-balanced in that it preserves the steady state at rest. Here, we restrict ourselves to the one-dimensional case to introduce the basic principles and to better analyze the details of the algorithm. The extension of the scheme to two space dimensions will be left to a sequel study. Starting from the governing equations, the Runge-Kutta discontinuous Galerkin method is introduced in the next section. On this basis, a detailed description of the new wetting and drying treatment is given in Section 3, which is verified in several test cases in Section 4. The paper closes with a final discussion and conclusions in Section 5.

2. The shallow water equations and their RKDG discretization

The one-dimensional shallow water model is defined by two equations, the first stating the conservation of mass and the second describing the balance of forces in form of the momentum equation. The system can be written in the compact conservative form

\[ \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}) \]  

where the vector of unknowns is given by \( \mathbf{U} = (h, hu)^T \). Here and below, we have written the partial derivatives with respect to time \( t \) and space \( x \) as indices, i.e., \( \mathbf{U}_t = \frac{\partial \mathbf{U}}{\partial t} \). The quantity \( h = h(x, t) \) denotes the fluid depth of a uniform density water layer and \( u = u(x, t) \) is the depth-averaged horizontal particle velocity. The flux function is defined by \( \mathbf{F}(\mathbf{U}) = (hu, hu^2 + \frac{1}{2}gh^2)^T \), where \( g \) is the gravitational constant. Furthermore, the bathymetry (bottom topography) \( b = b(x) \) is represented by the source term \( \mathbf{S}(\mathbf{U}) = (0, -ghb_x)^T \).

In order to obtain realistic simulation results in certain cases, further source terms have to be introduced into the equations. One of these is bottom friction which can highly affect the wetting and drying behavior of the flow. However, here we restrict ourselves to the inviscid case and the inclusion of further source terms is left to future research. A simple numerical treatment of bottom friction in case of discontinuous Galerkin methods can be found in Kesserwani & Liang [15], which is based on a split-implicit discretization.

For the discretization the governing equations are solved on the domain \( [x_{\min}, x_{\max}] \), which is divided into intervals (cells) \( I_i = (x_{i-1/2}, x_{i+1/2}) \). On each interval, the equations (1) are multiplied by a test function \( \varphi \) and integrated. Integration by parts of the flux term leads to the weak DG formulation

\[
\int_{I_i} \varphi \mathbf{U}_t \, dx - \int_{I_i} \varphi_x \mathbf{F}(\mathbf{U}) \, dx + \left[ \varphi \mathbf{F}^*(\mathbf{U}) \right]_{x_{i+1/2}}^{x_{i-1/2}} = \int_{I_i} \varphi \mathbf{S}(\mathbf{U}) \, dx .
\]  

Note that the interface flux \( \mathbf{F}^* \) is not defined in general, since the solution can have different values at the interface in the adjacent cells. This problem is circumvented in the discretization by using the (approximate) solution of the corresponding Riemann problem. For the simulations in this study we used the Rusanov solver [29], but other Riemann solvers such as HLLE [8] gave similar results.

System (2) is further discretized using a semi-discretization in space with a piecewise polynomial ansatz for the discrete solution components and test functions \( \varphi_k \). To obtain formally second-order accuracy, we use piecewise linear functions, which are represented by nodal Lagrange basis functions [10, 13], the latter being also employed as test functions. In view of a two-dimensional extension of the scheme, \( n \)-point Gauss-Legendre quadrature is applied to obtain an (exact) discretization of the integral terms. In each cell \( I_i \), this discretization in space leads to a system of ordinary differential equations (ODEs) for the local vector of degrees of freedom \( \tilde{\mathbf{U}}_i(t) \), where \( \mathbf{U}(x, t) = \sum_j (\tilde{\mathbf{U}}_i)_j \varphi_j \) for \( x \in I_i \). It is of the form

\[
\frac{\partial \tilde{\mathbf{U}}_i}{\partial t} = \int_{I_i} \left( \varphi_x \mathbf{F}(\mathbf{U}) + \varphi \mathbf{S}(\mathbf{U}) \right) \, dx - \left[ \varphi \mathbf{F}^*(\mathbf{U}) \right]_{x_{i+1/2}}^{x_{i-1/2}} = : \mathbf{H}_h(\tilde{\mathbf{U}}_i),
\]  

where \( \varphi = \mathbf{M}^{-1} \varphi_k \), and \( \mathbf{M} \) is the local mass matrix with \( M_{jk} = \int_{I_i} \varphi_j \varphi_k \, dx \). The resulting (global) system for the degrees of freedom of all cells \( \tilde{\mathbf{U}}_h \) is then solved using a total-variation diminishing (TVD)
s-stage Runge-Kutta scheme [11, 24], which is of the general form

$$
\tilde{U}_h^{(0)} = \tilde{U}_h^n
$$

$$
\tilde{U}_h^{(p)} = \Pi_h \left\{ \sum_{q=0}^{p-1} \alpha_{pq} \tilde{U}_h^{(q)} + \beta_{pq} \Delta t^n H_h \left( \tilde{U}_h^{(q)} \right) \right\} \quad \text{for } p = 1 \ldots s
$$

$$
\tilde{U}_h^{n+1} = \tilde{U}_h^{(s)}
$$

with Runge-Kutta coefficients $\alpha_{pq}$ and $\beta_{pq}$ and a time step size $\Delta t^n = t^{n+1} - t^n$. $H_h(\tilde{U}_h)$ represents the right hand side of the ODE (3) extended to all cells. The limiter $\Pi_h$, which is applied in each Runge-Kutta stage $p$ to obtain the intermediate solution $\tilde{U}_h^{(p)}$, is usually employed to stabilize the scheme in case of discontinuities. However, as stated above it can be also used for dealing with the problem of wetting and drying. In the remainder of this article, we use Heun’s method, which is the second-order representative of a standard Runge-Kutta TVD scheme. This completes the description of the Runge-Kutta discontinuous Galerkin (RKDG) method.

For (2), exact quadrature rules are a basic requirement for well-balancing [30]. At the cell interfaces no problems occur, since we use a continuous representation for the bottom topography. Otherwise, one can use the technique introduced in [30, 32], which is based on hydrostatic reconstruction of the interface values [1] and adds a higher order correction to the source term.

3. Wetting and drying algorithm

When it comes to wetting and drying, i.e., parts of the domain have water depth $h = 0$, several problems arise which must be handled by the numerical algorithm. First, the wet-dry transition might be within a cell and cannot be exactly represented by a piecewise (smooth) polynomial DG discretization (cf. Fig. 1). The result is the occurrence of artificial gradients in the surface elevation that can influence the tendencies of the momentum equation and render the scheme unbalanced. Furthermore, one must ensure that the fluid depth remains non-negative. Otherwise the shallow water equations are undefined at these points. The third and in the authors’ opinion least investigated problem is that near the wet/dry interface, both, fluid depth and momentum go to zero, which yields an ill-conditioned computation of the velocity $u = (hu)/h$ in these regions.

In the following, it is first described how to deal with the occurrence of artificial gradients at the wet/dry interface. After that a new limiter is introduced, which prevents the fluid depth from being negative and controls the ill-conditioned computation of the velocity through the adjustment of the momentum variable.
3.1. The wet/dry interface

There are flow configurations in which the DG discretization does not represent the physical situation accurately near the wet/dry interface. According to Bates & Hervouet [3] (see also [9]) we distinguish between two general situations, which will be referred to as of “flooding”-type and “dam-break”-type (Fig. 1). In the flooding-type situation the water level may rise by inflow from the deep water. This situation also includes the still water lake at rest, where no change occurs. If the exact shore line lies within a cell, an unphysical surface elevation gradient may evolve, as can be seen in the left part of Fig. 1. In the “dam-break” situation the inflow water level is higher than the dry point, like in the event of an upstream dam-break. There are two different configurations, which are displayed in the middle and on the right of Fig. 1. These situations are discretized physically correct.

Therefore, we only modify the computation of the momentum tendencies in the flooding-type situation. Such cells can be identified by comparing the maximum value of the surface elevation $H = h + b$ with the maximum in bottom topography $b$ within each cell, i.e., for cell $I_i$ we check if

$$\max_{x \in I_i} H(x) - \max_{x \in I_i} b(x) < \text{TOL}_{\text{wet}},$$

where $\text{TOL}_{\text{wet}}$ is a tolerance for the water depth under which it is considered dry. Since in the given piecewise linear configuration the extreme values are attained in the two vertices (Lagrange nodes) of a cell, it suffices to check the condition in these two points.

In the flooding-type cells the inner flux and source terms due to gravitational forces are neglected, which is equivalent to setting $g$ equal to zero in these terms. The interface flux at the dry node is zero due to zero fluid depth. However, the flux term at the wet interface has to be considered, since it is also present for the adjacent wet cell. In order to be well-balanced in the semi-dry cell, we add a flux term including only the gravitational part computed on the basis of the fluid depth from the semi-dry cell at the wet interface. For a configuration as in Fig. 1 (left) – assuming that this is cell $I_i$ – the momentum equation in this cell then reads

$$\int_{I_i} \varphi(hu)_i \, dx - \int_{I_i} \varphi_x \left( hu^2 + \frac{g}{2} h^2 \right) \, dx + \int_{I_i} \varphi_h \varphi h b_i \, dx = 0,$$

where $(\varphi h^2)(x_{i-1/2},+) = (\varphi h^2)(x_{i-1/2},-) + q \varphi h^2)I_{i-1/2,+})$ of the still water steady state this ensures that all flux terms vanish and the momentum tendency is zero.

We stress that this flux correction is only needed for well-balancing. In case of the non-still water case it is not necessary, but numerical evidence shows that there is no substantial difference between the two options, either.

3.2. Velocity based “limiting” of momentum

While limiting was originally introduced to obtain stable computations involving shocks, here we employ this numerical technique to stably discretize the wet/dry interface. Since the limiting modifies the solution itself in a DG method, and not only the values for the computation of the interface fluxes as in finite volume methods, further care must be taken for well-balancing. Therefore, we require that the limiter does not alter the steady state of the lake at rest. This is ensured by the Barth-Jespersen limiter [2] with limiting in hydrostatic variables, i.e., in $H = h + b$, which will be used throughout this work. This limiter essentially limits the in-cell distribution such that it does not exceed the minimum and maximum cell mean values of the surrounding cells. As we will see in the test cases, numerical evidence shows that limiting in fluid depth $h$ improves the drying process at the coast in some situations considerably. As an alternative we propose a blending between the two limiting procedures in $H$ and in $h$ to balance the different requirements for the scheme.
Concerning the positivity of the fluid depth, Xing et al. [32] have shown that a fluid depth, which is initially positive in certain quadrature points, leads to positive mean values provided a suitable CFL condition is met. For piecewise linear polynomials the quadrature points of the trapezoidal rule are relevant. These are the nodal values in each cell. The resulting CFL condition to ensure positivity in the mean is \( \frac{c_{\text{max}} \Delta t}{\Delta x} \leq 1/2 \), where \( c_{\text{max}} = \max \{|u| + \sqrt{gh}\} \). This condition is less restrictive than the CFL condition for linear stability of the DG method, the latter being \( \frac{c_{\text{max}} \Delta t}{\Delta x} \leq 1/3 \) for piecewise linear polynomials. Positivity of the whole distribution is then obtained by scaling around the cell mean values. Further details can be found in [32].

The linear momentum distribution is "limited" by analyzing the resulting velocity distribution. This provides a stable computation near the wet/dry interface in the situation when both, \( h \) and \( hu \) get small. As we will see, the modified momentum can have a higher in-cell variation in some cases.

We start the description of the limiter by specifying the limiting process in \( H = h + b \) (resp. \( h \)) on a nodal basis. Let \( H_{i-1/2,+} = H_i(x_{i-1/2}) \) and \( H_{i+1/2,-} = H_i(x_{i+1/2}) \) be the nodal values based on the linear distribution within cell \( I_i \), and define the cell mean value and variation by
\[
\overline{H}_i = \frac{H_{i+1/2,-} + H_{i-1/2,+}}{2} \quad \text{and} \quad \Delta H_i = \frac{H_{i+1/2,-} - H_{i-1/2,+}}{2}.
\]
Then in a nodal based form the limiting in \( H \) can be computed from a limited in-cell variation, which is given by
\[
\Delta H_i^{\text{lim}} = \begin{cases} 
\text{sign}(\Delta H_i) \cdot \min\{|\Delta H_i|, |\overline{H}_{i+1} - \overline{H}_i|, |\overline{H}_i - \overline{H}_{i-1}|\} & \text{if } (\overline{H}_{i+1} - \overline{H}_i)(\overline{H}_i - \overline{H}_{i-1}) > 0, \\
0 & \text{otherwise}.
\end{cases}
\]
The limited nodal values are
\[
H_{i-1/2,+}^{\text{lim}} = H_{i-1/2,+} + (\Delta H_i - \Delta H_i^{\text{lim}}) \quad \text{and} \quad H_{i+1/2,-}^{\text{lim}} = H_{i+1/2,-} + (\Delta H_i^{\text{lim}} - \Delta H_i).
\]
The limited values for the fluid depth are obtained from \( h_{\text{lim,final}}^{\text{final}} = H_{i+1/2,+}^{\text{lim}} - b_{i+1/2,+} \). Positivity is ensured by the application of the positivity preserving limiter of Xing et al. [32] in a subsequent step.

This procedure can be applied in the same way to \( h \), resulting in different, unbalanced in-cell distributions, but it can improve the drying process as stated above. Therefore, we introduce an alternative limiting procedure. Since both results from either limiting in \( H \) or \( h \) are linear and preserve the mean state, any convex linear combination does so, as well. We define a blending parameter, which linearly varies between 0 and 1 for minimum in-cell surface elevations \( H_{\text{min}} \) between the minimum and maximum in-cell bathymetry values \( b_{\text{min}} \) and \( b_{\text{max}} \). It is given by
\[
\lambda = \max\left(0, \min\left(1, \frac{H_{\text{min}} - b_{\text{min}}}{b_{\text{max}} - b_{\text{min}}} \right)\right)
\]
and the final nodal values are computed by
\[
h_{i\pm 1/2,+}^{\text{lim,final}} = \lambda (H_{i\pm 1/2,+}^{\text{lim}} - b_{i\pm 1/2,+}) + (1 - \lambda) b_{i\pm 1/2,+}.
\]

For the velocity based "limiting" of the momentum variable let \( u_i = (hu)_i \) be the velocity resulting from the mean values and \( u_{i\pm 1/2,+} = (hu_i(x_{i\pm 1/2})/h_i(x_{i\pm 1/2}) \) the velocities in the vertices based on the distributions of cell \( I_i \). Then the minimum and maximum velocities computed from the mean values of the surrounding cells are denoted by
\[
\begin{align*}
u_i^{\text{min}} &= \min\{u_{i-1}, u_i, u_{i+1}\} \quad \text{and} \quad u_i^{\text{max}} = \max\{u_{i-1}, u_i, u_{i+1}\},
\end{align*}
\]
Each vertex value is limited to be within the bounds of \( u_{i-1/2,+}^{\text{lim}} \) and \( u_{i+1/2,-}^{\text{lim}} \):

\[
u_{i-1/2,+}^{\text{lim}} = \max\{\min\{u_{i-1/2,+}, u_{i+1/2,-}^{\text{max}}\}, u_{i-1/2,+}^{\text{min}}\}
\]

\[
u_{i+1/2,-}^{\text{lim}} = \max\{\min\{u_{i+1/2,-}, u_{i+1/2,-}^{\text{max}}\}, u_{i+1/2,-}^{\text{min}}\}
\]

and the resulting velocity in the respective other vertex is computed by assuming a linear momentum distribution with prescribed mean value and a (fixed) linear distribution of the fluid depth:

\[
u_{i-1/2,+}^{\text{lim}} = \frac{(hu)_{i-1/2,+} - (hu)_{i-1/2,+}^{\text{lim}}}{h_{i-1/2,+}}
\]

\[
u_{i+1/2,-}^{\text{lim}} = \frac{(hu)_{i-1/2,+} - (hu)_{i-1/2,+}^{\text{lim}}}{h_{i-1/2,+}}
\]

Finally, the momentum distribution with the smallest in-cell velocity variation is chosen. We set

\[
\Delta m_{i-1/2}^{\text{lim}} = \begin{cases} 
    \nu_{i+1/2,-}^{\text{lim}} \cdot h_{i-1/2,+}^{\text{lim}} - (hu)_{i-1/2,+}, & \text{if } \left|\nu_{i+1/2,-}^{\text{lim}} - \nu_{i+1/2,-}^{\text{lim}}\right| < \\
    (hu)_{i-1/2,+} - \nu_{i-1/2,-}^{\text{lim}} \cdot h_{i-1/2,+}^{\text{lim}}, & \text{otherwise}
\end{cases}
\]

and compute the nodal values of the momentum by

\[
(hu)_{i-1/2,+}^{\text{lim}} = (hu)_{i-1/2,+} + (\Delta m_{i} - \Delta m_{i}^{\text{lim}}) ,
\]

\[
(hu)_{i+1/2,-}^{\text{lim}} = (hu)_{i+1/2,-} + (\Delta m_{i}^{\text{lim}} - \Delta m_{i}) ,
\]

where \( \Delta m_{i} = (hu)_{i+1/2,-} - (hu)_{i-1/2,+} \). For further stabilization we set the momentum to zero in all nodes where the fluid depth drops under the wet tolerance:

\[
(hu)_{i+1/2,\mp}^{\text{lim,final}} = \begin{cases} 
    0, & \text{if } h_{i+1/2,\mp}^{\text{lim}} \leq \text{TOL}_{\text{wet}} \\
    (hu)_{i+1/2,\mp}^{\text{lim}}, & \text{otherwise}
\end{cases}
\]

To illustrate the velocity based limiting it is visualized in Fig. 2 where we have compiled a possible configuration in surface elevation and momentum at the wet/dry interface in the top row. The resulting velocity distribution is displayed in the rightmost figure. As one can see, the velocity in the center cell has an unphysical extreme value at \( x = 2 \), where both, fluid depth and momentum become small. For the limiting, the cell mean values of \( h \) and \( (hu) \) are computed (magenta diamonds) and upper and lower limits for the velocity are derived (dashed magenta line). This results in two limited velocity and associated momentum distributions, which are marked as red with triangles and green with squares at the end points. The final distribution is the velocity distribution with the smallest in-cell variation – in this case it is the red one with triangles. As one can see the associated momentum distribution has a slightly bigger in-cell variation compared to the original one.

4. Numerical results

In this section the presented numerical scheme is tested against different configurations with wet/dry interfaces. Some address the well-balancing property of the method such as the classical “lake at rest”, others elaborate on the stability such as the oscillatory flow in a parabolic basin. A more realistic situation is simulated by a long-wave runup onto a beach, which resembles the arrival of a tsunami at the coast. In all simulations the gravitational constant is set to \( g = 9.81 \). Here and below we omit the dimensions of the physical quantities, which should be thought in the standard SI system with m (meter), s (seconds) etc. as
basic units. The discrete initial conditions and the bottom topography are derived from the analytical ones by interpolation at the nodal (cell interface) points. The wet tolerance is set to \( TOL_{\text{wet}} = 10^{-8} \), and we use limiting in \( H = h + b \) if not stated otherwise. As stated above, a CFL number \( cfl \leq 1/3 \) results in a time step \( \Delta t = cfl \Delta x/c_{\text{max}} \) which provides a stable and positive solution. In the following simulations, we always choose a fixed time step which satisfies this stability constraint. Besides fluid depth and momentum we also often show the velocity, which is derived by the quotient of the two other quantities.

4.1. Lake at rest

To verify the well-balancing property of the scheme, a basin is setup with a fluid at rest and an initially horizontal surface elevation. The test domain \([0, 1]\) has periodic boundary conditions. In the middle of the domain is an island, which is defined by the (smooth) bottom topography

\[
b_s(x) = b(r) = \begin{cases} 
a \cdot \frac{\exp(-0.5/(r_m^2 - r^2))}{\exp(-0.5/r_m^2)} & \text{if } r < r_m, \\
0 & \text{otherwise}
\end{cases}
\]

where \( r = |x - 0.5| \), and the parameters are set to \( a = 1.2 \) and \( r_m = 0.4 \). The initial fluid depth is \( h(x, 0) = \min(0, 1 - b(x)) \) (see Fig. 3 left). The initial momentum is set to \( (hu)(x, 0) = 0 \), which means that the fluid is in steady state. The domain is discretized into 50 cells, and the timestep is set to 0.002. This corresponds to a CFL number of 0.3. The solution is integrated over 10 000 timesteps until \( t_{\text{max}} = 20 \).
Figure 3: Initial surface elevation for the “lake at rest” test case with smooth (left) and noisy (right) bathymetry. Depicted are the bathymetry (gray dashed) and total water depth (red solid).

Figure 4: Errors in fluid depth (left) and momentum (right) over time for the “lake at rest” test case measured in the $L^\infty$ norm. Simulation with smooth bathymetry (black) and with noisy bathymetry (gray).

As in case of the exact solution, the discrete initial conditions should be preserved and only small deviations due to numerical truncation errors should occur. In Fig. 4 the errors in fluid depth and momentum are displayed over time using the maximum norm (black lines). It can be seen that for both variables only errors within the range of machine accuracy develop.

A second run was conducted in which some “noise” was added to the bottom topography defined above in form of high frequency data. It is defined by

$$b_n(x) = b_s(x) + \sum_{j=1}^{3} a_j \sin(16j\pi x + p_j)$$

where the amplitudes are given by $a = (0.1, 0.2, 0.3)$ and the phase shifts by $p = (1.6, 3.2, 0.5)$. The resulting initial condition is shown in Fig. 3 (right). The errors from the computed solution over time (Fig. 4 gray lines) stay also within the range of machine accuracy, although they are a bit larger than in the smooth case.

4.2 Small perturbation over under-resolved bathymetry

To further study the well-balancing property of the method, a small perturbation is added to a fluid at rest. During the experiment, one of the two resulting waves should travel over an exponential bottom topography which is touching the fluid surface only in one point. This numerically challenging test case was proposed by Xing et al. [32]. The domain $[-5, 5]$ is given with a bottom topography $b(x) = 0.5 \exp(-10x^2)$, and the initial condition (see Fig. 5) is defined by

$$h(x, 0) = \begin{cases} 
0.5 - b(x) + 0.0001 & \text{if } -3 \leq x \leq -2, \\
0.5 - b(x) & \text{otherwise,}
\end{cases} \quad \text{and} \quad u(x, 0) \equiv 0.$$
From the zoomed-in version of the initial data one can see that the tip of the bottom topography becomes singular for coarse grid resolutions. The two waves emerging from the perturbation travel at characteristic speeds $\pm \sqrt{gh}$ to the left and right direction.

In Fig. 6 snapshots of the surface elevation at times $0.8, 1.6$ and $2.4$ are displayed for different numerical resolutions. Due to the singularity of the bottom tip the flow develops some artificial oscillations when the right going wave travels over the tip. This happens especially at the relatively coarse resolution of 250 uniform grid cells. As the resolution becomes finer, the artificial oscillations vanish, which can be seen for the grids with 1250 and 6250 uniform cells. Such oscillatory behavior was also observed by Xing et al. [32] when applying their limiter to the surface elevation $H$.

### 4.3. Riemann problems with wet/dry fronts

Riemann problems are usually considered to assess the shock-capturing capabilities of a scheme. Furthermore, they provide an analytical solution to compare with. Here, we setup two different Riemann problems which include dry areas and have been used before to evaluate similar inundation schemes [5, 6, 32] to demonstrate the positivity-preserving capability of the methods. Both problems have a flat bottom ($b \equiv 0$).

The first considered test is a dam break problem with fluid on the left side and a dry area on the right side. The computational domain is $[-300, 300]$ with transparent boundary conditions. The initial conditions are

$$h(x, 0) = \begin{cases} 10 & \text{if } x \leq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad u(x, 0) \equiv 0.$$  

The analytical solution of this problem is a rarefaction wave and can be found in [5].

For the discretization the domain is divided into 200 uniform intervals. The timestep is 0.05 resulting in a maximum CFL number of 0.32 during the simulation. We should note that in this case the maximum CFL number with respect to the fluid velocity (i.e., computed with $c_{\text{max}} = \max \{ |u| \}$) is 0.31. This value is achieved in the simulations in the vicinity of the wet/dry interface. Therefore, this test case drives the inundation scheme to its stability limits. The solution (cell mean values) is displayed at times $t = 4, 8$ and 12 in Fig. 7 together with the analytical solution. Besides the fluid depth and the momentum we have also plotted the resulting velocity distribution. The exact solution is well approximated by the simulation results and no instabilities occur. Only a small lag of the wet/dry interface in the numerical solution compared to the exact one can be observed, which can be best seen from the velocity distributions.

In the second Riemann problem the domain $[-200, 400]$ is initially wet everywhere with initial conditions

$$h(x, 0) = \begin{cases} 5 & \text{if } x \leq 0, \\ 10 & \text{otherwise,} \end{cases} \quad \text{and} \quad u(x, 0) = \begin{cases} 0 & \text{if } x \leq 0, \\ 40 & \text{otherwise.} \end{cases}$$
Due to the drying condition $\sqrt{gh_l} + u_l < -\sqrt{gh_r} + u_r$, a dry region emerges for $t > 0$, and two expansion waves traveling into opposite directions occur. The analytical solution to this problem can be also found in [5].

The domain is discretized into 200 uniform cells and the timestep is 0.02, which results in a CFL number 0.33. The cell mean values of the simulation are compared to the exact solution in Fig. 8 at times $t = 2, 4$ and 6. Also in this case the analytical solution is well captured. The only problem occurs in the drying area, where the discrete fluid depth does not get below the wet tolerance, resulting in an artificial velocity profile there.

4.4. Oscillatory flow in a parabolic bowl

A numerically challenging test goes back to Thacker [27], where an oscillatory flow in a domain with parabolic bottom topography is considered. Even the analytical solution for the nonlinear shallow water equations is known in this case. It has become another standard test problem for inundation modeling and has been applied to several schemes [e.g. 15, 32]. On the domain $[-5000, 5000]$ the bottom topography is defined by $b(x) = h_0(x/a)^2$, where $a = 3000$ and $h_0 = 10$ define the shape of the parabolic basin. Note that the boundary conditions for the domain are irrelevant since the boundary is in the dry part of the solution. The analytical solution of the water surface is then given by

$$ h(x,t) + b(x) = h_0 - \frac{B^2}{4g} (1 + \cos(2\omega t)) - \frac{Bx}{2a}\sqrt{\frac{8h_0}{g}} \cos \omega t, $$

Figure 6: Small perturbation over under-resolved bathymetry. Numerical results of surface elevation (red solid) at times 0.8, 1.6 and 2.4 (from left to right) for grid resolutions of 250 cells (top), 1250 cells (middle) and 6250 cells (bottom).
Figure 7: Numerical and exact solution of the dam break problem at times $t = 4, 8$ and $12$. Cell mean values of fluid depth (top left), momentum (top right) and velocity (bottom). The exact solution is given by the black solid line.

Figure 8: Numerical and exact solution of the Riemann problem with two expansion waves at times $t = 2, 4$ and $6$. Cell mean values of fluid depth (top left), momentum (top right), velocity (bottom). The exact solution is given by the black solid line.
where we set \( \omega = \sqrt{2gh_0/a} \) and \( B = 5 \). Furthermore, the velocity in the wet part is given by

\[
u(x,t) \equiv \frac{Ba_\omega}{\sqrt{2gh_0}} \sin \omega t.
\]

Thus, the solution involves a periodical movement of the wet/dry interface at both sides of the basin. The initial conditions are shown in Fig. 9.

The domain is discretized using 200 uniform cells, and a timestep of 1.0 is used, which approximately corresponds to a maximum CFL number of 0.3. Also in this case we obtain a high maximum CFL number with respect to the fluid velocity, which is 0.29. This value is again achieved in the vicinity of the wet/dry interfaces, but this time on the side where the fluid recedes.

The simulation is executed until \( t_{\max} = 3000 \), when the flow has oscillated a bit more than two periods. The numerical solution compared to the exact one is shown in Fig. 10 at times 1000, 2000, 3000. Fluid depth and momentum are well approximated by the DG scheme and indistinguishable from the analytical solution. Only in the velocity field deviations are visible in the vicinity of the wet/dry interface and mostly in case of receding fluid (drying). In these cases a thin film of fluid is left for some time steps and the division of small numbers to compute the velocity leads to these numerical errors. Despite everything, these deviations stay bounded.

These artifacts become smaller if blended limiting is done as described in Section 3.2. This can be seen in Fig. 11. Limiting in \( h \) clearly improves the solution in the vicinity of receding fluid. The discrete solutions for fluid depth and momentum (not shown) are almost indistinguishable from the exact one also in this case.

The test case of an oscillatory flow in a parabolic bowl is also suitable to evaluate the conservation of mass and of total energy

\[
E = \int_{\Omega} h u^2 / 2 + gh(h/2 + b) \, dx
\]

for the numerical method, since there is no flow across the boundary of the domain. In Fig. 12 we have plotted the relative mass error and change in total energy with respect to the initial data over time. It can be seen that the mass error is in the range of machine accuracy and there are also only very small fluctuations in total energy.

Another question is the order of convergence of the method. While we cannot expect second order accuracy due to the non-smooth transition between wet and dry regions in the flow variables, the accuracy should be at least around one. For the convergence calculation we have computed the solution up to \( t = 1000 \) on several grids with cells ranging from 50 to 3200 and fixed CFL number. The experimental convergence rate is calculated by the formula

\[
\gamma_c := \frac{\log(||e_c||/||e_f||)}{\log(\Delta x_c/\Delta x_f)}.
\]

In this definition, \( e_c \) and \( e_f \) are the computed error functions of the solution on a coarse and a fine grid (denoted by the number of cells) and \( \Delta x_c \) and \( \Delta x_f \) are the corresponding grid spacings. The results are
Figure 10: Free surface elevation (left), momentum (middle) and velocity (right) of an oscillatory flow in a parabolic bowl. Numerical (red solid) and exact solution (blue dash-dotted) at times $t = 1000$, $t = 2000$, $t = 3000$ (from top to bottom). Wet/dry tolerance is set to $10^{-8}$. Limiting in surface elevation $H$.

shown in Fig. 13 and Table 1 for the $L^2$ and $L^\infty$ norms. It can be seen that in the $L^2$ norm the mean convergence rate is close to $1.5$ for both, fluid depth and momentum. In the $L^\infty$ the rates are around $1.0$. These results are consistent with the literature [cf. 6]. The low values in the $L^\infty$ norm can be probably explained by the errors arising in the zone of receding water, leading to a small phase error that spoils the maximum norm, but is rather harmless in the two-norm.

We varied the wet tolerance $TOL_{wet}$ for this test case to see how much the stability of the scheme depends on it. Interestingly, we could vary it over a broad range from $10^{-14}$ to $10^0$ and did not even get any stability issues. The parameter merely affected the accuracy of the solution. For $TOL_{wet} = 10^0 = 1$, when the parameter is of the order of the solution, large deviations from the exact solution become visible. On the other hand, for a value of $10^{-2}$ the errors in the velocity field vanish for the most part as can be seen in Fig. 14.

4.5. Tsunami runup onto a sloping beach

A more realistic test case is given by the propagation of a tsunami wave onto a uniformly sloping beach. It was originally defined as a benchmark problem in [28]. Besides the slope of the beach the initial surface elevation (Fig. 15) and momentum with $(hu)(x,0) \equiv 0$ is given. The solution is sought on the domain $[-500, 50000]$ and the bottom topography is set to $b(x) = 5000 - 0.1x$. At the right boundary of the domain a simple transparent boundary condition is implemented. However, the crucial task is to correctly simulate the inundation process on the interval $[-400, 800]$. The analytical solution at times $t = 160$, $175$ and $220$ can be derived by the initial-value-problem technique introduced by Carrier et al. [7] and can be found in [28].
Figure 11: Velocity of an oscillatory flow in a parabolic bowl. Numerical (red solid) and exact solution (blue dash-dotted) at times $t = 1000$, $t = 2000$, $t = 3000$ (from left to right). Wet/dry tolerance is set to $10^{-8}$. Blended limiting.

Figure 12: Relative mass error (left) and relative change in total energy (right) of the numerical solution simulating the oscillatory flow in a parabolic bowl.

In the presented simulation, the domain is discretized into 1010 uniform cells and the timestep is 0.05, which approximately corresponds to a CFL number of 0.22 at the deepest point (right side) of the domain. The results are displayed for the inundation zone in Fig. 16, where the fluid depth can also be compared to the analytical solution. At the first observation time $t = 160$ the water recedes, whereas $t = 175$ is the reversal point between drainage and flooding. At the final observation $t = 220$ the coast is still flooded. As in the previous test cases, the analytical fluid depth is well approximated by the discretization. Some spurious velocity deviations can be seen especially in the drying process ($t = 160, 175$), but these are bounded and do not grow over time.

For comparison of the two different limiting procedures, the results for the velocity with blended limiting is shown in Fig. 17 (as opposed to limiting in surface elevation in Fig. 16). In this case the velocity distributions show slightly more deviations near the wet/dry interface. However, the maximum CFL number with respect to the fluid velocity is much smaller in this test case due to the large fluid depth at the wet end of the domain.

5. Conclusions

In this work a wetting and drying treatment has been proposed for piecewise linear discontinuous Galerkin discretizations of the one-dimensional shallow water equations. It features a velocity based “limiting” of the momentum variable which ensures the schemes’ stability in the vicinity of wet/dry interfaces. The non-destructive limiting of steady states at rest together with a flux modification of semi-dry cells result in a well-balanced method. Several test cases verified the applicability of the scheme to a variety of flow regimes. They show that the scheme is well-balanced, mass conservative and stable for rapid transitions of the edge.
Figure 13: Error of the oscillatory flow in a parabolic bowl test case at $t = 1000$ in the $L^2$ (blue circles) and $L^\infty$ (green triangles) norms for different grid sizes. Left: fluid depth, right: momentum.

![Graph showing error vs. grid size](image)

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<th>$L^2(m)$</th>
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Table 1: Convergence rates between different grid levels for the oscillatory flow in a parabolic bowl for fluid depth ($h$) and momentum ($m$) in the $L^2$ and $L^\infty$ norms. Also displayed is the mean convergence rate $\gamma_{fitted}$, which is obtained by a least squared fit.

Figure 14: Velocity of an oscillatory flow in a parabolic bowl. Numerical (red solid) and exact solution (blue dash-dotted) at times $t = 1000$, $t = 2000$, $t = 3000$ (from top to bottom). Wet/dry tolerance is set to $10^{-2}$. Limiting in surface elevation $H$.

![Graph showing velocity vs. time](image)
Figure 15: Tsunami runup onto a sloping beach. Initial surface elevation at $t = 0$.

Figure 16: Tsunami runup onto a sloping beach. Computed surface elevation (left), momentum (middle) and velocity (right) from the DG method (red solid) compared to the exact solution (blue dash-dotted, only for surface elevation) at times $t = 160$ (top), $t = 175$ (middle), $t = 220$ (bottom). Limiting in surface elevation $H$. 
of the water body. Furthermore, the experimental order of convergence is approximately 1.5 in the $L^2$ norm and 1 in the maximum norm. The scheme has only one parameter, which is the wet tolerance, under which a node value is considered to be dry. Computations suggest that the scheme is robust with respect to the particular value of this parameter. Two different limiting strategies were developed in this work: Either by limiting only in the surface elevation $H = h + b$ as it is done in hydrostatic reconstruction methods, or by blending limiting in $H$ and the fluid depth $h$, the latter resulting in more stable computations in case of rapid wetting and drying. It still has to be investigated, which strategy is better for practical computations.

Beside the major goal of the development of a robust inundation scheme we achieved a simple and straightforward algorithmic structure, which should make it possible to apply the proposed treatment to other DG models. The method is implemented into a second order Runge-Kutta scheme by only a small flux modification and the implementation of the new limiter. The need for positivity, well-balancedness and stability were clearly addressed by specific components of the algorithm.

The development of limiters for discontinuous Galerkin methods is still under heavy development. The velocity based “limiting” of the momentum resembles the limiting procedure in finite volume methods of other than the primary flow variables (like limiting in primitive or characteristic variables when otherwise working in conservative variables). We are optimistic that the general concept proposed in this work might be transferable to other problems such as DG methods for the solution of inviscid compressible flow applications.

Concerning the wetting and drying treatment our target applications are tsunami and storm surge simulations. In this respect the extension of the algorithm to the two-dimensional case is ongoing research. Furthermore, possibilities to extend the proposed concept to higher than linear discontinuous Galerkin elements are investigated.

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